ON THE INTERSECTION OF TWO PLANE CURVES

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1. Introduction and Statement of Results

The following question was raised and partially answered by Geng Xu in [X].

Question 1.1. *Let D be a general degree d curve in $\mathbb{P}^2$. What is the minimal number $i(d, m)$ of points in the set-theoretical intersection $C \cap D$ for any degree m irreducible curve C (suppose that C and D meet properly)?*

This problem is related to a conjecture of Kobayashi and Zaidenberg which states that for a sufficiently general curve $D \subset \mathbb{P}^2$ of degree $d \geq 5$, as general in the sense that D lies in $|\mathcal{O}_{\mathbb{P}^2}(d)| \cong \mathbb{P}^{d(d+3)/2}$ with countably many closed proper subvarieties removed, the affine variety $\mathbb{P}^2 \backslash D$ is hyperbolic. One necessary condition for $\mathbb{P}^2 \backslash D$ being hyperbolic is that there is no rational curve $C \subset \mathbb{P}^2$ meeting D set-theoretically at fewer than three points; otherwise, there is going to be a nonconstant holomorphic map $\mathbb{C} \to C \backslash (C \cap D) \subset \mathbb{P}^2 \backslash D$. This property of $\mathbb{P}^2 \backslash D$ was called "algebraic hyperbolic" in [DSW].

Definition 1.1. A quasi-projective variety is called *algebraic hyperbolic* if it does not contain a curve whose normalization is an elliptic curve or a rational curve with two points removed, i.e., $\mathbb{P}^1 \backslash \{p, q\} \cong \mathbb{C}^* \cong \operatorname{Spec} \mathbb{C}[x, x^{-1}]$.

Obviously, hyperbolicity implies algebraic hyperbolicity for smooth quasi-projective varieties.

Using an elegant deformation-theoretical argument, Xu proved the following [X, Theorem 1].

Theorem 1.1 (Xu). *For $d \geq 3$, $\min_{m>0} i(d, m) = d - 2$.*

He thus concluded that every curve $C \subset \mathbb{P}^2$ meets D at no less than three distinct points and hence $\mathbb{P}^2 \backslash D$ is algebraic hyperbolic for a sufficiently general curve D of degree $d \geq 5$. This bound is sharp for $m = 1$ and it is achieved by a bitangent or flex line to D.

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The purpose of this paper is two-fold. First, we will try to sharpen his bound with both d and m fixed. Second, we will try to extend his result to other surfaces.

By dimension count, one may expect that $i(d,m) = dm - r_{d,m}$ where $r_{d,m}$ is the dimension of the linear series cut out on D by all curves of degree m, namely, $r_{d,m} = m(m+3)/2$ for $m < d$ and $r_{d,m} = dm - (d-1)(d-2)/2$ for $m \geq d$. However, this is simply false for $m \geq d \geq 3$ by the following construction.

Let L be a bitangent (or flex) line to D. Since D is general, L meets D at $d-2$ distinct points. Let $L(X,Y,Z)$ and $D(X,Y,Z)$ be the homogeneous defining equations of L and D, respectively. Then for any degree $m-d$ homogeneous polynomial $G(X,Y,Z)$, $L^m(X,Y,Z) + D(X,Y,Z)G(X,Y,Z) = 0$ defines a degree m curve C which meets D at $d-2$ points, which are the intersections between L and D. If we choose $G(X,Y,Z)$ general enough, C is irreducible and actually smooth. Hence, by Xu's result, $i(d,m) = d-2$ for $m \geq d \geq 3$.

Nevertheless, we think that $i(d,m)$ has the expected value for $d > m$, i.e.,

Conjecture 1.1. *For $d > m$ and $d \geq 3$, $i(d,m) = dm - r_{d,m}$.*

Although we cannot prove the above conjecture, we have the following estimate for $i(d,m)$ when $m < d$.

Theorem 1.2. *For $d > m$,*

$$i(d,m) \geq \min\left(dm - \frac{m(m+3)}{2}, 2dm - 2m^2 - 2\right).$$

An easy corollary of the above theorem is the following

Corollary 1.1. *For $2d \geq 3m-2$ and $d \geq 3$, $i(d,m) = dm - m(m+3)/2$, i.e., Conjecture 1.1 holds for $2d \geq 3m-2$. In particular, it holds for $m \leq 4$.*

In order to formulate Kobayashi type conjectures on surfaces other than $\mathbb{P}^2$, we need to study Question 1.1 in the following general setting.

Question 1.2. *Let S be a smooth surface and let L and M be two line bundles on S. Let D be a general member of $|L|$. What is the minimal number $i(L,M)$ of points in the set-theoretical intersection $C \cap D$ for any irreducible curve $C \in |M|$ (suppose that C and D meet properly)?*

We will work on rational ruled surfaces, although our method can be extended to other surfaces. By convention, let $\mathbb{F}_n$ be the rational ruled surface given by $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ over $\mathbb{P}^1$ and let C and F be the zero section and the fiber of $\mathbb{F}_n \to \mathbb{P}^1$, i.e., $C^2 = -n$, $C \cdot F = 1$ and $F^2 = 0$. We have the following lower bound for $i(L,M)$ with L ample.

Theorem 1.3. *Let $L = \mathcal{O}(aC + bF)$ be an ample line bundle on $\mathbb{F}_n$ with $a \geq 2$ and $b \geq 2$. Then $\min_M i(L, M) = \min(a-1, b-an, b-n-1)$, where M runs over all line bundles with irreducible general global sections.*

It follows immediately from Theorem 1.3 that every curve on $\mathbb{F}_n$ meets D at no less than three distinct points for a sufficiently general $D \in |aC + bF|$ with $a \geq 4$ and $b \geq \max(4 + n, 3 + an)$. Therefore

Corollary 1.2. *For a sufficiently general curve $D \in |aC + bF|$ on $\mathbb{F}_n$ with $a \geq 4$ and $b \geq \max(4 + n, 3 + an)$, the complement $\mathbb{F}_n \backslash D$ is algebraic hyperbolic.*

Notice that the bound in Theorem 1.3 can be achieved by a curve in $|C|$ or $|F|$, which is necessarily a rational curve. So the lower bounds for a and b in the above corollary cannot be improved.

This enables us to formulate Kobayashi conjecture on $\mathbb{F}_n$.

Conjecture 1.2 (Kobayashi Conjecture on Rational Ruled Surfaces)**.** *For a sufficiently general curve $D \in |aC + bF|$ on $\mathbb{F}_n$ with $a \geq 4$ and $b \geq \max(4 + n, 3 + an)$, the complement $\mathbb{F}_n \backslash D$ is hyperbolic.*

The organization of this paper is as follows. Theorem 1.2 and 1.3 will be proved in the next two sections, respectively. At the end of the third section, we will also discuss some related problems.

Throughout this paper we work exclusively over the field of complex numbers $\mathbb{C}$.

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2. Proof of Theorem 1.2

Let $W_\delta \subset |\mathcal{O}(m)| \times |\mathcal{O}(d)|$ be the incidence correspondence defined by

$$\begin{aligned} W_\delta = \{(C, D) : C \in |\mathcal{O}(m)| \text{ is irreducible}, D \in |\mathcal{O}(d)| \text{ is smooth}, \\ \text{and } C \text{ and } D \text{ meet set-theoretically at } \delta \text{ points}\}. \end{aligned}$$

Our proof of Theorem 1.2 is carried out by estimating the dimension of W_δ and show that $\dim W_\delta < \dim |\mathcal{O}(d)|$ if $\delta < dm - m(m+3)/2$ and $\delta < 2dm - 2m^2 - 2$ and hence it cannot dominate $|\mathcal{O}(d)|$ in this case.

Let $\pi : W_\delta \to |\mathcal{O}(m)|$ be the projection of W_δ to $|\mathcal{O}(m)|$ and let C be a general point of $\pi(W_\delta)$ (by a general point, we mean a general point of an irreducible component of $\pi(W_\delta)$).

Let π_C be the fiber of $\pi : W_\delta \to |\mathcal{O}(m)|$ over C and let (C, D) be a general point on $\pi(C)$. There exists a series of blowups of $\mathbb{P}^2$ such that

the proper transforms $\widetilde{C}$ and $\widetilde{D}$ of C and D meet at smooth points on both curves (since we assume that D is smooth, we only have to resolve the singularities of C where D passes through). Let $\widetilde{\mathbb{P}^2}$ be the resulting blowup of $\mathbb{P}^2$ and E_i $(1 \le i \le \alpha)$ be the exceptional divisors. Suppose that $\widetilde{C} \in |\mathcal{O}(mH - \sum_{i=1}^{\alpha} r_i E_i)$ and $\widetilde{D} \in |\mathcal{O}(dH - \sum_{i=1}^{\alpha} E_i)|$, where $r_i > 1$ and H is the pull-back of the hyperplane divisor of $\mathbb{P}^2$.

Suppose that $\widetilde{C}$ and $\widetilde{D}$ meet at points $p_1, p_2, ..., p_\beta$ with multiplicities $m_1, m_2, ..., m_\beta$, respectively, where $\beta \le \delta \le \alpha + \beta$. Then by a deformation-theoretical argument, the tangent space $T_{\pi_C,(C,D)}$ of π_C at (C, D) is contained in

$$\begin{aligned}
&H^0(\mathcal{O}_{\widetilde{D}}(dH - \sum_{i=1}^{\alpha} E_i) \otimes \mathcal{O}_{\widetilde{D}}(-\sum_{j=1}^{\beta}(m_j - 1)p_j))\\
&= H^0(\mathcal{O}_{\widetilde{D}}((d-m)H + \sum_{i=1}^{\alpha}(r_i - 1)E_i) \otimes \mathcal{O}_{\widetilde{D}}(\sum_{j=1}^{\beta} p_j)).
\end{aligned}$$

Hence by Riemann-Roch

$$\begin{aligned}
\dim \pi_C &\le h^0(\mathcal{O}_{\widetilde{D}}((d-m)H + \sum_{i=1}^{\alpha}(r_i - 1)E_i + \sum_{j=1}^{\beta} p_j))\\
&= \frac{d(d+3)}{2} + \sum_{i=1}^{\alpha}(r_i - 1) + \beta - dm\\
&+ h^0(\mathcal{O}_{\widetilde{D}}((m-3)H - \sum_{i=1}^{\alpha}(r_i - 1)E_i) \otimes \mathcal{O}_{\widetilde{D}}(-\sum_{j=1}^{\beta} p_j)).
\end{aligned}$$

It is not hard to see that

$$\begin{aligned}
&h^0(\mathcal{O}_{\widetilde{D}}((m-3)H - \sum_{i=1}^{\alpha}(r_i - 1)E_i) \otimes \mathcal{O}_{\widetilde{D}}(-\sum_{j=1}^{\beta} p_j))\\
&\le h^0(\mathcal{O}_{\widetilde{C}}((m-3)H - \sum_{i=1}^{\alpha}(r_i - 1)E_i) \otimes \mathcal{O}_{\widetilde{C}}(-\sum_{j=1}^{\beta} p_j))\\
&+ \sum_{i=1}^{\alpha} \frac{(r_i - 1)(r_i - 2)}{2}.
\end{aligned} \tag{2.1}$$

This can be shown by the following argument.

We further blow up $\widetilde{\mathbb{P}^2}$ at points $p_1, p_2, ..., p_\beta$ with corresponding exceptional divisors $F_1, F_2, ..., F_\beta$. We still denote the resulting surface

by $\widetilde{\mathbb{P}^2}$ and the proper transforms of C and D by $\widetilde{C}$ and $\widetilde{D}$. We have the following exact sequence on $\widetilde{\mathbb{P}^2}$

$$\begin{aligned}
0 &\to H^0((m-d-3)H - \sum_{i=1}^{\alpha}(r_i-2)E_i) \\
&\to H^0((m-3)H - \sum_{i=1}^{\alpha}(r_i-1)E_i - \sum_{j=1}^{\beta} F_j) \\
&\to H^0(\mathcal{O}_{\widetilde{D}}((m-3)H - \sum_{i=1}^{\alpha}(r_i-1)E_i - \sum_{j=1}^{\beta} F_j)) \\
&\to H^1((m-d-3)H - \sum_{i=1}^{\alpha}(r_i-2)E_i).
\end{aligned}$$

Obviously, $h^0((m-d-3)H - \sum_{i=1}^{\alpha}(r_i-2)E_i) = 0$ and

$$\begin{aligned}
&h^2((m-d-3)H - \sum_{i=1}^{\alpha}(r_i-2)E_i) \\
&= h^0((d-m)H + \sum_{i=1}^{\alpha}(r_i-1)E_i + \sum_{j=1}^{\beta} F_j) \\
&= \frac{(d-m)(d-m+3)}{2} + 1.
\end{aligned}$$

Hence by Riemann-Roch,

$$h^1((m-d-3)H - \sum_{i=1}^{\alpha}(r_i-2)E_i) = \sum_{i=1}^{\alpha} \frac{(r_i-1)(r_i-2)}{2}.$$

Therefore

(2.2)

$$\begin{aligned}
&h^0(\mathcal{O}_{\widetilde{D}}((m-3)H - \sum_{i=1}^{\alpha}(r_i-1)E_i - \sum_{j=1}^{\beta} F_j)) \\
&\le h^0((m-3)H - \sum_{i=1}^{\alpha}(r_i-1)E_i - \sum_{j=1}^{\beta} F_j) + \sum_{i=1}^{\alpha} \frac{(r_i-1)(r_i-2)}{2}.
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
&h^0(\mathcal{O}_{\widetilde{C}}((m-3)H - \sum_{i=1}^{\alpha}(r_i-1)E_i - \sum_{j=1}^{\beta} F_j)) \\
&= h^0((m-3)H - \sum_{i=1}^{\alpha}(r_i-1)E_i - \sum_{j=1}^{\beta} F_j).
\end{aligned} \tag{2.3}
$$

Combining (2.2) and (2.3), we obtain (2.1). Therefore

$$
\begin{aligned}
\dim \pi_C \le{}& \frac{d(d+3)}{2} + \sum_{i=1}^{\alpha} \frac{r_i(r_i-1)}{2} + \beta - dm \\
&+ h^0(\mathcal{O}_{\widetilde{C}}((m-3)H - \sum_{i=1}^{\alpha}(r_i-1)E_i) \otimes \mathcal{O}_{\widetilde{C}}(-\sum_{j=1}^{\beta} p_j)) \\
={}& \frac{d(d+3)}{2} + \sum_{i=1}^{\alpha} \frac{r_i(r_i-1)}{2} + \beta - dm + h^0(\omega_{\widetilde{C}} \otimes \mathcal{O}_{\widetilde{C}}(-\sum_{j=1}^{\beta} p_j))
\end{aligned}
$$

where $\omega_{\widetilde{C}}$ is the dualizing sheaf of $\widetilde{C}$.

By Clifford's theorem (see for example [ACGH, pp. 107-8]), we have either

$$
h^0(\omega_{\widetilde{C}} \otimes \mathcal{O}_{\widetilde{C}}(-\sum_{j=1}^{\beta} p_j)) = 0
$$

or

$$
h^0(\mathcal{O}_{\widetilde{C}}(\sum_{j=1}^{\beta} p_j)) \le \beta/2 + 1.
$$

Hence correspondingly, we have either

$$
\dim \pi_C \le \frac{d(d+3)}{2} + \sum_{i=1}^{\alpha} \frac{r_i(r_i-1)}{2} + \beta - dm
$$

or

$$
\dim \pi_C \le \frac{d(d+3)}{2} + \frac{\beta}{2} - dm + \frac{(m-1)(m-2)}{2}.
$$

Since C is a general member of $\pi(W_\delta)$ and C has singularities with multiplicities r_i ($1 \le i \le \alpha$), by Zariski's theorem on the deformation of planary curve singularities [Z], we have

$$
\dim \pi(W_\delta) \le \frac{m(m+3)}{2} - \sum_{i=1}^{\alpha} \frac{r_i(r_i-1)}{2}.
$$

And hence we have either

$$\dim W_\delta \le \frac{d(d+3)}{2} + \beta - dm + \frac{m(m+3)}{2}$$

or

$$\dim W_\delta \le \frac{d(d+3)}{2} + \frac{\beta}{2} - dm + m^2 + 1 - \sum_{i=1}^{\alpha} \frac{r_i(r_i-1)}{2}.$$

Therefore, if W_δ dominates $|\mathcal{O}(d)|$, we necessarily have

$$\delta \ge \beta \ge \min\left(dm - \frac{m(m+1)}{2}, 2(dm - m^2 - 1)\right).$$

This finishes the proof of Theorem 1.2.

3. Intersections of Two Curves on Rational Ruled Surfaces

Our approach to Theorem 1.3 is different from that of Xu's. A key ingredient of Xu's proof of Theorem 1.1 is a map from the deformation space of the pair (D, E), where $D \in |\mathcal{O}_{\mathbb{P}^2}(d)|$ and $E \in |\mathcal{O}_{\mathbb{P}^2}(m)|$ meet at no less than s distinct points, to the cohomology group of a sheaf over D. More specifically, let (Z_0, Z_1, Z_2) be generic homogeneous coordinates of $\mathbb{P}^2$ and let $F_0 \in H^0(\mathcal{O}_{\mathbb{P}^2}(d))$ and $G_0 \in H^0(\mathcal{O}_{\mathbb{P}^2}(m))$ be the defining equations of D and E. A first order deformation of (D, E) is given by $F_1 \in H^0(\mathcal{O}_{\mathbb{P}^2}(d))$ and $G_1 \in H^0(\mathcal{O}_{\mathbb{P}^2}(m))$ such that the curves $\{F_0 + tF_1 = 0\}$ and $\{G_0 + tG_1 = 0\}$ meet at no less than s points over the ring $\mathbb{C}[t]/(t^2)$. It is observed by Xu that [X, Lemma 1]

(3.1)

$$\frac{\partial F_0}{\partial Z_i} G_1 - \frac{\partial G_0}{\partial Z_i} F_1 \in H^0(D, \mathcal{O}_D(d+m-1) \otimes \mathcal{O}_D(-\sum_{j=1}^{s} (\mu_j - 1)p_j))$$

for $i = 0, 1, 2$, where D and E meet at $p_1, p_2, ..., p_s$ with multiplicities $\mu_1, \mu_2, ..., \mu_s$, respectively.

The relation (3.1) forms the basis of Xu's proof of Theorem 1.1. If we were to prove Theorem 1.3 following Xu's line of argument, we would have to come up with a relation similar to (3.1) on $\mathbb{F}_n$, which we are unable to do. So we find that Xu's analysis, though ingenious on its own, is hard, if not impossible, to carry out on surfaces other than $\mathbb{P}^2$. Therefore, we will adopt a different approach to Theorem 1.3, which is based upon degeneration and induction.

Let Δ be a disk parameterized by t and let $Y \subset \mathbb{F}_n \times \Delta$ be a pencil of curves in $|aC + bF|$ whose central fiber $Y_0 = G \cup \Gamma$ is reducible with two components $G \in |(a-1)C + (b-n-1)F|$ and $\Gamma \cong \mathbb{P}^1 \in |C + (n+1)F|$. Let $X \subset \mathbb{F}_n \times \Delta$ be a family of curves on $\mathbb{F}_n$ whose general fiber X_t

$(t \neq 0)$ meets Y_t at s distinct points (a base change may be needed to ensure the existence of X). If X_0 meets Y_0 properly, we may deduce $s \geq \min(a-1, b-an, b-n-1)$ by the induction hypothesis that

$$\begin{aligned}\#(X_0 \cap G) &\geq \min(a-2, b-an-1, b-n-2) \\ &= \min(a-1, b-an, b-n-1) - 1\end{aligned}$$

and by the fact that $\#(X_0 \cap \Gamma) \geq 1$, where we use the notation $\#(A \cap B)$ to denote the number of points in the set-theoretical intersection $A \cap B$ between the two curves A and B. Of course, some care has to be taken to make sure that X_0 meets Γ at at least one point outside of $G \cap \Gamma$ (see below). Unfortunately, X_0 may very well contain G or Γ as a component. So we have to regard $|\mathcal{O}_{Y_0}(X_0)|$ as the limit linear series $\lim_{t\to 0} |\mathcal{O}_{Y_t}(X_t)|$ and, correspondingly, $Y_0 \cap X_0$ as the limit of the section $Y_t \cap X_t$ in $|\mathcal{O}_{Y_t}(X_t)|$. For an introduction to the theory of limit linear series, please see, for example, [E-H] or [H, Chap. 5].

For the purpose of induction, we will prove Theorem 1.3 in the following slightly more general form.

Proposition 3.1. *Let $L = \mathcal{O}(aC + bF)$ be an ample line bundle on $\mathbb{F}_n$ with $a \geq 2$ and $b \geq 2$. Then for a sufficiently general curve $D \in |L|$,*

1. *$\#(D \cap E) \geq \min(a-1, b-an, b-n-1)$ for any curve $E \subset \mathbb{F}_n$ that meets D properly;*
2. *in addition, there exists a set Σ_D consisting of countably many points on D such that if $\#(D \cap E) = \min(a-1, b-an, b-n-1)$ for some E, $(D \cap E) \subset \Sigma$.*

We prove Proposition 3.1 by induction on $\min(a-1, b-an, b-n-1)$.

If $\min(a-1, b-an, b-n-1) = 1$, we only need to verify the second part of the proposition. Notice that D has genus $g(D) = 1 + \frac{1}{2}(a-2)(b-an) + \frac{1}{2}a(b-n-2) \geq 1$. If D meets E at a single p for some E, $\mathcal{O}_D(\mu p) = \mathcal{O}_D(E)$, where $\mu = D \cdot E$. If we fix the divisor class of E, there are only finitely many points p with this property since $g(D) \geq 1$. Therefore, there are only countably many points p such that $D \cap E = \{p\}$ for some E.

Suppose that $\min(a-1, b-an, b-n-1) \geq 2$. Notice that $\mathcal{O}((a-1)C + (b-n-1)F)$ is ample under this assumption.

Let X, Y, G and Γ be defined as before. Suppose that G and Γ meet at points $p_1, p_2, ..., p_l$, where $l = a + b - n - 2$. Let $M = \mathcal{O}(X_t)$ be the line bundle associated to X_t.

Let $\sigma_t \in |\mathcal{O}_{Y_t}(X_t)|$ be the section cut out by X_t on Y_t and let $\sigma_0 = \lim_{t\to 0} \sigma_t$. Let $\sigma_\Gamma = \sigma_0|_\Gamma$ and $\sigma_G = \sigma_0|_G$ be the restrictions of σ_0 to Γ and G, respectively. Then σ_Γ is a section in

$$|\mathcal{O}_\Gamma(\mu(p_1 + p_2 + ... + p_l)) \otimes M| = |\mathcal{O}_\Gamma(\mu G) \otimes M|$$

and σ_G is a section in

$$|\mathcal{O}_G(-\mu(p_1 + p_2 + \dots + p_l)) \otimes M| = |\mathcal{O}_G(-\mu\Gamma) \otimes M|$$

where μ is an integer and σ_Γ and σ_G are cut out by sections in $|\mathcal{O}(\mu G) \otimes M|$ and $|\mathcal{O}(-\mu\Gamma) \otimes M|$, respectively.

Suppose that $\mathcal{O}(-\mu\Gamma) \otimes M$ is nontrivial. Then by induction hypothesis σ_G vanishes at no less than $\min(a-2, b-an-1, b-n-2)$ distinct points. If σ_Γ vanishes at at least one point other than $p_1, p_2, \dots, p_l$, we are done; if not, we have either $\mathcal{O}(\mu G) \otimes M$ is trivial and σ_Γ is nowhere vanishing or σ_Γ only vanishes at $p_1, p_2, \dots, p_l$.

If $\mathcal{O}(\mu G) \otimes M$ is trivial and σ_Γ is nowhere vanishing, then for any two points among $p_1, p_2, \dots, p_l$, say p_1 and p_2, the ratio $\sigma_G(p_1)/\sigma_G(p_2)$ is uniquely determined by the choice of the pencil Y. Actually we have the following very explicit relation

$$\frac{\sigma_G(p_1)}{\sigma_G(p_2)} = \left(\frac{f(p_1)}{f(p_2)}\right)^{-\mu} \tag{3.2}$$

where $f \in |L|$ is the section which cuts out a general member Y_t of the pencil Y. If σ_G vanishes at more than $\min(a-2, b-an-1, b-n-2)$ distinct points, there is nothing to prove; otherwise, σ_G vanishes at exactly $\min(a-2, b-an-1, b-n-2)$ distinct points. Then by induction hypothesis, there are only countably many possible choices of σ_G. However, by (3.2), the ratio $\sigma_G(p_1)/\sigma_G(p_2)$ can be made into an arbitrary complex value by a choice of f (and thus a choice of the pencil Y). Contradiction.

If σ_Γ only vanishes at $p_1, p_2, \dots, p_l$, since we have already taken care of the case that $\mathcal{O}(\mu G) \otimes M$ is trivial and σ_Γ is nowhere vanishing, we may assume that σ_Γ vanishes at at least one point among $p_1, p_2, \dots, p_l$, say p_1. Then σ_G must vanish at p_1 as well. Again, if σ_G vanishes at more than $\min(a-2, b-an-1, b-n-2)$ distinct points, there is nothing to prove; otherwise, σ_G vanishes at exactly $\min(a-2, b-an-1, b-n-2)$ distinct points. By induction hypothesis, $p_1 \in \Sigma_G$. But if we choose Γ generically, $p_1 \notin \Sigma_G$. Contradiction.

Now suppose that $\mathcal{O}(-\mu\Gamma) \otimes M$ is trivial. If $\sigma_G = 0$, then σ_Γ vanishes at $p_1, p_2, \dots, p_l$ and $l = a+b-n-2 > \min(a-1, b-an, b-n-1)$; we are done. Otherwise, σ_G is no where vanishing. The ratio $\sigma_\Gamma(p_i)/\sigma_\Gamma(p_j)$ for any two points p_i and p_j among $p_1, p_2, \dots, p_l$, just as in (3.2), is uniquely determined by the choice of Y and is given by

$$\frac{\sigma_\Gamma(p_i)}{\sigma_\Gamma(p_j)} = \left(\frac{f(p_i)}{f(p_j)}\right)^{\mu}. \tag{3.3}$$

The rational map $|L| \to \mathbb{P}^{l-1}$ by sending $f \in |L|$ to

$$(f^\mu(p_1), f^\mu(p_2), ..., f^\mu(p_l)) \tag{3.4}$$

is dominant due to the facts that $H^0(\mathbb{F}_n, L)$ surjects onto $H^0(\Gamma, L)$ and $L \otimes \mathcal{O}_\Gamma(-\sum_{i\neq j} p_i)$ is base point free on Γ for each $1 \leq j \leq l$. On the other hand, the space

$$\{\sigma_\Gamma | \sigma_\Gamma \text{ vanishes at less than } l-1 \text{ distinct points}\}$$

has dimension $l-2$ and hence cannot dominate $\mathbb{P}^{l-1}$. So σ_Γ vanishes at at least $l-1 > \min(a-1, b-an, b-n-1)$ distinct points for a general choice of f by (3.3).

This finishes the proof of the first part of the proposition.

Suppose that σ_0 vanishes at exactly $\min(a-1, b-an, b-n-1)$ distinct points. This can happen only when $\mathcal{O}(-\mu\Gamma)\otimes M$ is nontrivial.

Suppose that σ_G vanishes at exactly $\min(a-2, b-an-1, b-n-2)$ distinct points. Our previous argument shows that σ_G does not vanish at $p_1, p_2, ..., p_l$ for a general choice of $G \cup \Gamma$. Then σ_Γ must vanish at a single point $p \notin \{p_1, p_2, ..., p_l\}$. Since $\#(G \cap \Gamma) \geq 2$, the natural map from $Y_0 \backslash \{p_1, p_2, ..., p_l\}$ to $\mathrm{Pic}(Y_0)$ is injective. So p is determined up to finitely many possibilities by M and the vanishing locus of σ_G. By induction, the vanishing locus of σ_G is contained in some countable set Σ_G depending only on G. So the vanishing locus of σ_0 is also contained in some countable set $\Sigma_{G\cup\Gamma}$ depending only on $G \cup \Gamma$.

Suppose that σ_G vanishes at exactly $\min(a-1, b-an, b-n-1)$ distinct points and suppose that there is a one-parameter family of $\sigma_0(u)$ with this property, where $\sigma_0(u)$ is parameterized by $u \in U$ for some irreducible curve U.

Suppose that $\mathcal{O}(\mu G) \otimes M$ is trivial. There exists $u_0 \in U$ such that $\sigma_G(u_0)$ vanishes at p_1. Since $\mathcal{O}(\mu G)\otimes M$ is trivial, $\sigma_\Gamma(u_0) = 0$ and hence $\sigma_G(u_0)$ vanishes at $p_1, p_2, ..., p_l$. But $l > \min(a-1, b-an, b-n-1)$. Contradiction.

Suppose that $\mathcal{O}(\mu G) \otimes M$ is nontrivial. Then $\sigma_\Gamma(u)$ vanishes at at least one point among $p_1, p_2, ..., p_l$, say p_1. Hence $\sigma_G(u)$ vanishes at p_1 for all $u \in U$. As u varies, another vanishing point of $\sigma_G(u)$ will approach p_1. So there exists $u_0 \in U$ such that $\sigma_G(u_0)$ vanishes at $\min(a-2, b-an-1, b-n-2)$ distinct points and among them vanishes at a general point p_1. Again this is impossible by induction. Contradiction.

This finishes the proof of Proposition 3.1.

The degeneration method we used can be applied to surfaces other than rational ruled surfaces. For example, we can give an alternative

proof of Xu's Theorem 1.1 by degenerating a degree d curve to a union of a degree $d-1$ curve and a line and arguing by induction.

A proof of Xu's Theorem 1.1 via degeneration. As in the case of Proposition 3.1, we need to add a clause to the theorem for the purpose of induction, i.e., we will prove the following statement by induction on d.

For a sufficiently general curve D of degree $d \geq 3$ in $\mathbb{P}^2$, $\#(D \cap E) \geq d-2$ for any curve $E \subset \mathbb{P}^2$ that meets D properly. In addition, there exists a set Σ_D of countably many points on D such that if $\#(D \cap E) = d-2$ for some E, $(D \cap E) \subset \Sigma_D$.

Let $Y \subset \mathbb{P}^2 \times \Delta$ be a pencil of degree d curves whose central fiber $Y_0 = G \cup \Gamma$ is the union of a curve G of degree $d-1$ and a line Γ and let $G \cap \Gamma = \{p_1, p_2, ..., p_l\}$ where $l = d-1$.

Let $X, M, \sigma_t, \sigma_0, \sigma_G, \sigma_\Gamma$ and μ be defined as before. Almost nothing in the argument of Proposition 3.1 needs changing except in the case that $\mathcal{O}(-\mu\Gamma) \otimes M$ is trivial and σ_G is nowhere vanishing. In this case, following our previous argument, we can show that σ_Γ vanishes at no less than $l-1$ points. The difference is that now we have $l-1 = d-2$ and we have to verify that there are only finitely many σ_Γ that vanishes at exactly $l-1$ distinct points. This is more or less obvious because the map from $|L|$ to $\mathbb{P}^{l-1}$ given by (3.4) is dominant and the space

$$\{\sigma_\Gamma | \sigma_\Gamma \text{ vanishes at exactly } l-1 \text{ distinct points}\}$$

has dimension $l-1$. $\square$

Our degeneration method also works for Del Pezzo surfaces.

Theorem 3.1. *Let $\widetilde{\mathbb{P}^2}$ be the blowup of $\mathbb{P}^2$ at $2 \leq r \leq 6$ general points and let $L_1, L_2, ..., L_k, ...$ be all the smooth rational curves on $\widetilde{\mathbb{P}^2}$ with self-intersection -1. Let L be an ample line bundle on $\widetilde{\mathbb{P}^2}$. Then for a sufficiently general curve $D \in |L|$,*

1. *$\#(D \cap E) \geq \min_k(D \cdot L_k)$ for any curve $E \subset \widetilde{\mathbb{P}^2}$ that meets D properly;*
2. *in addition, there exists a set Σ_D of countably many points on D such that if $\#(D \cap E) = \min_k(D \cdot L_k)$ for some E, $(D \cap E) \subset \Sigma_D$.*

Therefore, for a sufficiently general curve $D \in |L|$ with $\min_k(D \cdot L_k) \geq 3$, the complement $\widetilde{\mathbb{P}^2} \backslash D$ is algebraic hyperbolic.

Proof. Let $K_{\widetilde{\mathbb{P}^2}}$ be the canonical divisor of $\widetilde{\mathbb{P}^2}$. We argue by induction on $\min_k(D \cdot L_k)$.

For $\min_k(D \cdot L_k) = 1$, we need to verify that $g(D) \geq 1$, which is more or less obvious.

Suppose that $\min_k(D\cdot L_k) \geq 2$. Let $Y \subset \mathbb{P}^2 \times \Delta$ be a pencil of curves in $|L|$ whose central fiber $Y_0 = G \cup \Gamma$ is a union of $G \in \left|L \otimes \mathcal{O}(K_{\widetilde{\mathbb{P}^2}})\right|$ and $\Gamma \in \left|-K_{\widetilde{\mathbb{P}^2}}\right|$ and let $G \cap \Gamma = \{p_1, p_2, ..., p_l\}$.

Let $X, M, \sigma_t, \sigma_0, \sigma_G, \sigma_\Gamma$ and μ be defined as before. Again, the same argument for Proposition 3.1 goes through. We need only to check the following facts, all of which are routine exercises.

1. $l > \min_k(D \cdot L_k)$.
2. $H^0(\widetilde{\mathbb{P}^2}, L)$ surjects onto $H^0(\Gamma, L)$ and
$$L \otimes \mathcal{O}_\Gamma(-\sum_{i\neq j} p_i) = \mathcal{O}_\Gamma(-K_{\widetilde{\mathbb{P}^2}}) \otimes \mathcal{O}_\Gamma(p_j)$$
is base point free on Γ for each $1 \leq j \leq l$. Hence the map from $|L|$ to $\mathbb{P}^{l-1}$ given by (3.4) is dominant.
3. In the case that $\mathcal{O}(-\mu\Gamma) \otimes M$ is trivial and σ_G is nowhere vanishing, we can prove that σ_Γ vanishes at no less than $l-1$ distinct points as before. But actually, we can do better here since the space
$$\{\sigma_\Gamma | \sigma_\Gamma \text{ vanishes at less than } l \text{ distinct points}\}$$
has dimension $l-2$ due to the fact that Γ is elliptic instead of rational. Therefore, σ_Γ vanishes at no less than l distinct points.

□

When we go up in dimension, however, some essential difficulties present themselves. For example, in $\mathbb{P}^3$, fix a sufficient general surface S of degree d and it is expected that any curve meets S at no less than $d-4$ distinct points [X, Question 2]. Let Y be a pencil of degree d surfaces whose central fiber is a union of a degree $d-1$ surface and a plane and let X be family of curves in $\mathbb{P}^3$ meeting Y fiberwise. To carry out the argument as in dimension two, we need to take the limit $X_t \cap Y_t$ as an element in $A_0(Y_t)$, the 0-dimension Chow ring of Y_t. Of course, we do not know how to do this at present.

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